Varieties of affine modules

By B. CSÁKÁNY in Szeged

Let $\mathbf{R}$ be a ring with unit element. Any n-ary ($n>0$) polynomial $f=f(x_1, \ldots, x_n)$ of a unital right $\mathbf{R}$-module $\mathbf{A}$ is of the form

$$(1) \qquad x_1\varrho_1+\cdots+x_n\varrho_n$$

where $\varrho_i \in \mathbf{R}$ ($i=1, \ldots, n$). Denote by I the family of all polynomials of $\mathbf{A}$ satisfying $\sum_i \varrho_i=1$. We can associate with $\mathbf{A}$ the algebra $\mathbf{A}^*=\langle A; I\rangle$ which will be called an *affine module over* $\mathbf{R}$.

Affine modules were introduced by OSTERMANN and SCHMIDT in [9]; in the case when $\mathbf{R}$ is a field this notion coincides with that of the affine space over $\mathbf{R}$, treated by MAC LANE and BIRKHOFF in [2]. In [6] GIVANT, characterizing varieties in which all algebras are free, announced that all affine spaces over a division ring (defined similarly) form such a variety.

In what follows we show that all affine modules over any ring with unit element form a variety and we give an abstract characterization (in terms of subalgebras and congruences) up to (rational) equivalence in MAL'CEV's sense (see [3], Chapter 9) for varieties of affine modules. Such varieties over commutative rings as well as over fields are also characterized. Finally, we show that any variety of affine modules determines its "ring of scalars" up to isomorphism.

The basic terminology we use is adopted from [1]. Note, however, that we will denote polynomial symbols and polynomials induced by them in the same way. Sometimes we write $(\varrho_1, \ldots, \varrho_n)$ instead of the polynomial f given by (1). The base sets of algebras $\mathbf{A}, \mathbf{B}, \ldots$ will be denoted by $A, B, \ldots$. All rings which occur in the following are supposed to have a unit element and all modules will be unital right modules. We shall denote the class of all $\mathbf{R}$-modules by $\mathscr{M}(\mathbf{R})$, and the class of all affine modules over $\mathbf{R}$ by $\mathscr{A}(\mathbf{R})$. For any $\mathbf{R}$-module $\mathbf{B}$ the associated affine module will always be denoted by $\mathbf{B}^*$.

Proposition. *For any ring* $\mathbf{R}$, $\mathscr{A}(\mathbf{R})$ *is a variety.*

Proof. $\mathbf{S}(\mathscr{A}(\mathbf{R}))\subseteq\mathscr{A}(\mathbf{R})$. Let $\mathbf{A}$ be an $\mathbf{R}$-module, and suppose that $\mathbf{M}$ is a subalgebra of $\mathbf{A}^*$. Choose an $s\in M$. We show that $M_s=\{m-s\mid m\in M\}$ is the base set of a submodule $\mathbf{M}_s$ of $\mathbf{A}$, and $\mathbf{M}\cong\mathbf{M}_s^*\in\mathscr{A}(\mathbf{R})$. Indeed, if $a, b\in M$, $\varrho\in R$, then $(a-s)+(b-s)=(1, -1, 1)(a, s, b)-s\in M_s$ and $(a-s)\varrho=(\varrho, -\varrho, 1)(a, s, s)-s\in M_s$ hold. Further, let $m\varphi=m-s$ for any $m\in M$; then for arbitrary $\varrho_1, \ldots, \varrho_n\in R$ (with sum 1) and $m_1, \ldots, m_n\in M$ we have

$$(\varrho_1, \ldots, \varrho_n)(m_1\varphi, \ldots, m_n\varphi) = \sum_i \varrho_i m_i - \sum_i \varrho_i s = \sum_i \varrho_i m_i - s =$$

$$= \big((\varrho_1, \ldots, \varrho_n)(m_1, \ldots, m_n)\big)\varphi,$$

hence φ is an isomorphism of $\mathbf{M}$ onto $\mathbf{M}_s^*$.

$\mathbf{H}(\mathscr{A}(\mathbf{R}))\subseteq\mathscr{A}(\mathbf{R})$. If θ is a congruence relation (shortly: congruence) on $\mathbf{A}^*$, then θ is a congruence on $\mathbf{A}$, too: $a\equiv a_1$, $b\equiv b_1$ (θ) $(a, a_1, b, b_1\in A)$ imply $a+b=(1, -1, 1)(a, 0, b)\equiv(1, -1, 1)(a_1, 0, b_1)=a_1+b_1(\theta)$ and $a\varrho=(\varrho, 1-\varrho)(a, 0)\equiv(\varrho, 1-\varrho)(a_1, 0)=a_1\varrho(\theta)$ for any $\varrho\in R$. Hence $\mathbf{A}^*/\theta\cong(\mathbf{A}/\theta)^*$ follows immediately.

$\mathbf{P}(\mathscr{A}(\mathbf{R}))\subseteq\mathscr{A}(\mathbf{R})$, since $\Pi(\mathbf{A}_j^*\mid j\in J)\cong(\Pi(\mathbf{A}_j\mid j\in J))^*$. This completes the proof.

Theorem 1. *A variety $\mathscr{R}$ is equivalent to $\mathscr{A}(\mathbf{R})$ for some ring $\mathbf{R}$ if and only if*

(∗) *in any algebra of $\mathscr{R}$ every subalgebra is a block of a unique congruence, and every block of any congruence is a subalgebra.*

Proof. Observe first, that condition (∗) means exactly that $\mathscr{R}$ is Hamiltonian (i.e., subalgebras are blocks of congruences; cf. [8]), idempotent (all operations are idempotent) and regular (see, e.g., [5]). Note that subalgebras and congruences are invariant under equivalence of varieties; hence to prove the necessity of condition (∗) it is enough to deal with the variety $\mathscr{A}(\mathbf{R})$ for an arbitrarily chosen ring $\mathbf{R}$.

Clearly, $\mathscr{A}(\mathbf{R})$ is idempotent. In order to show regularity we apply the characterization of regular varieties in [5]. As for the operation $(1, -1, 1)$, the identity $(1, -1, 1)(x, x, z)=z$ and the identical implication $(1, -1, 1)(x, y, z)=z\Rightarrow x=y$ are fulfilled, $\mathscr{A}(\mathbf{R})$ is regular. Finally, if $\mathbf{A}$ is an $\mathbf{R}$-module and $\mathbf{M}$ is a subalgebra of $\mathbf{A}^*$, then — as we have seen in the preceding proof — M is a block of a congruence of $\mathbf{A}$. By the definition of operations, all congruences of $\mathbf{A}$ are also congruences of $\mathbf{A}^*$. Thus M is a block of a congruence of $\mathbf{A}^*$; hence $\mathscr{A}(\mathbf{R})$ is Hamiltonian.

The proof of sufficiency needs the following lemma which is analogous to Lemma 2 in [4]:

Lemma 1. *Let $\mathscr{R}$ be a Hamiltonian, idempotent and regular variety. Suppose $\mathbf{A}\in\mathscr{R}$; let $\mathbf{B}$, $\mathbf{C}$ be subalgebras of $\mathbf{A}$ such that $B\cup C$ generates $\mathbf{A}$ and $B\cap C$ consists of a single element e. Then $\mathbf{A}\cong\mathbf{B}\times\mathbf{C}$ and there exists an isomorphism $\varphi\colon\mathbf{A}\to\mathbf{B}\times\mathbf{C}$ such that for any $b\in B$ and $c\in C$, $b\varphi=\langle b, e\rangle$ and $c\varphi=\langle e, c\rangle$ hold.*

Outline of proof. The Hamiltonian property and regularity of $\mathcal{R}$ imply that **B** and **C** determine uniquely the congruences β, resp. γ of **A**, for which they are blocks. For any $a \in A$ there exist a unique $b \in B$ and a unique $c \in C$ with $a \equiv b(\gamma)$ and $a \equiv c(\beta)$. The mapping φ defined by $a\varphi = (b, c)$ is the desired isomorphism of **A** onto $\mathbf{B} \times \mathbf{C}$. Details — *mutatis mutandis* — may be found in [4].

We prove the sufficiency by constructing a suitable ring **R** for any Hamiltonian, idempotent and regular variety $\mathcal{R}$ such that $\mathcal{A}(\mathbf{R})$ will be equivalent to $\mathcal{R}$. We establish the equivalence of these classes with the use of the following fact: varieties $\mathcal{K}$ and $\mathcal{L}$ are equivalent if and only if there exists a weak isomorphism (in the sense of GOETZ [7]) between the countably generated free algebras of these classes, which induces a one-to-one correspondence between free generating sets. To make so, we need a further lemma.

Lemma 2. *Let* **R** *be an arbitrary ring. If* **F** *is a free algebra over* $\mathcal{M}(\mathbf{R})$ *with the free generating set* X, *then* $\mathbf{F}^*$ *is a free algebra over* $\mathcal{A}(\mathbf{R})$ *with the free generating set* $\{0\} \cup X$.

Proof. Suppose that in $\mathbf{F}^*$ the equality

$$(\varrho_1, \dots, \varrho_m)(y_{i_1}, \dots, y_{i_m}) = (\sigma_1, \dots, \sigma_n)(y_{j_1}, \dots, y_{j_n}) \tag{2}$$

holds, where $\varrho_k, \sigma_l \in R$, $\sum_k \varrho_k = \sum_l \sigma_l = 1$, and $y_{i_k}, y_{j_l} \in \{0\} \cup X$. We may assume that the y_{i_k} as well as the y_{j_l} are pairwise different. It is enough to prove that (2) holds identically in $\mathcal{A}(\mathbf{R})$.[1] If none of y_{i_k} and y_{j_l} equals $0 (\in F)$, then we may consider (2) as an equality in **F**, which holds identically in $\mathcal{M}(\mathbf{R})$, and hence in $\mathcal{A}(\mathbf{R})$, too.

Suppose now, e.g., $y_{i_1} = 0$. If none of y_{j_l} equals 0, then $y_{i_2}\varrho_2 + \dots + y_{i_m}\varrho_m = y_{j_1}\sigma_1 + \dots + y_{j_n}\sigma_n$ in **F**, and this equality holds identically in any **R**-module. Consider **R** as an **R**-module and substitute $1 (\in R)$ for each y_{i_k} and y_{j_l} in this equality. Then we get $\varrho_2 + \dots + \varrho_m = \sigma_1 + \dots + \sigma_n (=1)$, whence $\varrho_1 = 0$. Hence it follows that (2) holds identically in $\mathcal{A}(\mathbf{R})$.

In the remainder case, let $y_{j_l} = 0$ for some l $(1 \leq l \leq n)$. Now the above consideration shows that $y_{i_2}\varrho_2 + \dots + y_{i_m}\varrho_m = y_{j_1}\sigma_1 + \dots + y_{j_{l-1}}\sigma_{l-1} + y_{j_{l+1}}\sigma_{l+1} + \dots + y_{j_n}\sigma_n$ holds identically in any **R**-module and $\varrho_1 = \sigma_l$. Therefore, $a\varrho_1 + a_{i_2}\varrho_2 + \dots + a_{i_m}\varrho_m = a_{j_1}\sigma_1 + \dots + a_{j_{l-1}}\sigma_{l-1} + a\sigma_l + a_{j_{l+1}}\sigma_{l+1} + \dots + a_{j_n}\sigma_n$ for arbitrary elements a, a_{i_k}, a_{j_l} (with $a_{i_k} = a_{j_l}$ whenever $y_{i_k} = y_{j_l}$) of any **R**-module. This means that (2) holds identically in $\mathcal{A}(\mathbf{R})$, completing the proof of Lemma 2.

Let $\mathbf{F}_{012}$ be a free algebra over a Hamiltonian, idempotent and regular variety $\mathcal{R}$ (which will be fixed in the sequel) with the free generating set $\{x_0, x_1, x_2\}$. Let $\mathbf{F}_{01}$ and $\mathbf{F}_{02}$ denote the subalgebras of $\mathbf{F}_{012}$ generated by $\{x_0, x_1\}$ and $\{x_0, x_2\}$, respec-

[1]) This means: "if in (2) we replace $y_{i_1}, \dots, y_{j_n}$ by polynomial symbols $x_{i_1}, \dots, x_{j_n}$ respectively, and polynomials by the associated polynomial symbols, then we get an identity in $\mathcal{A}(\mathbf{R})$".

tively. Then $\mathbf{F}_{01}\cup\mathbf{F}_{02}$ generates $\mathbf{F}_{012}$. On the other hand, $\mathbf{F}_{01}\cap\mathbf{F}_{02}=\{x_0\}$. Indeed, let $x\in\mathbf{F}_{01}\cap\mathbf{F}_{02}$; then there exist binary polynomials g and h over $\mathbf{F}_{012}$ such that $x=g(x_0, x_1)=h(x_0, x_2)$. The second equality holds identically in $\mathscr{R}$; hence, by idempotency, we have $x=g(x_0, x_1)=h(x_0, x_0)=x_0$. Thus, we can apply Lemma 1: $\mathbf{F}_{012}\cong\mathbf{F}_{01}\times\mathbf{F}_{02}$, and there exists an isomorphism φ such that for any binary polynomial k over $\mathbf{F}_{012}$ $(k(x_0, x_1))\varphi=\langle k(x_0, x_1), x_0\rangle$ and $(k(x_0, x_2))\varphi=\langle x_0, k(x_0, x_2)\rangle$.

We can find a ternary polynomial f over $\mathbf{F}_{012}$ such that $(f(x_0, x_1, x_2))\varphi=\langle x_1, x_2\rangle$. Let $\mathbf{F}_\omega$ denote the free algebra over $\mathscr{R}$ with countable free generating set $\{x_0, x_1, \ldots\}$, and for any $x, y\in\mathbf{F}_\omega$, let $x+y=f(x_0, x, y)$. This binary algebraic function over $\mathbf{F}$ will be called addition. We show that $\langle F_\omega; +\rangle$ is an Abelian group.

First, $\langle x_1, x_2\rangle=(f(x_0, x_1, x_2))\varphi=f(x_0\varphi, x_1\varphi, x_2\varphi)=f(\langle x_0, x_0\rangle), \langle x_1, x_0\rangle, \langle x_0, x_2\rangle)=$ $=\langle f(x_0, x_1, x_0), f(x_0, x_0, x_2)\rangle$. Hence $f(x, y, x)=f(x, x, y)=y$ is an identity in $\mathscr{R}$. Then for any $x\in\mathbf{F}_\omega$ we get $x+x_0=f(x_0, x, x_0)=x$, and similarly, $x_0+x=x$; i.e., x_0 is the zero element for the addition.

Let now $\mathbf{F}_{n0n}$ be a free algebra over $\mathscr{R}$ with free generating set $\{y_1, \ldots, y_n, x_0, x_1, \ldots, x_n\}$. Let $\mathbf{F}_{n0}$ and $\mathbf{F}_{0n}$ denote the subalgebras of $\mathbf{F}_{n0n}$ generated by $\{x_0, x_1, \ldots, x_n\}$ and $\{x_0, y_1, \ldots, y_n\}$, respectively. Lemma 1 applied to algebras $\mathbf{F}_{n0n}$, $\mathbf{F}_{n0}$ and $\mathbf{F}_{0n}$ furnishes the following fact: $\mathbf{F}_{n0n}\cong\mathbf{F}_{n0}\times\mathbf{F}_{0n}$ and there exists an isomorphism ψ such that for any $(n+1)$-ary polynomial l over $\mathbf{F}_{n0n}$ $(l(x_0, x_1, \ldots, x_n))\psi=$ $=\langle l(x_0, x_1, \ldots, x_n), x_0\rangle$ and $(l(x_0, y_1, \ldots, y_n))\psi=\langle x_0, l(x_0, y_1, \ldots, y_n)\rangle$ hold. This implies

$$(3)\qquad p(f(x_0, x_1, y_1), \ldots, f(x_0, x_n, y_n))=f(x_0, p(x_1, \ldots, x_n), p(y_1, \ldots, y_n))$$

for any n-ary polynomial p over $\mathbf{F}_{n0n}$. Indeed,

$$\begin{aligned}
&p(f(x_0, x_1, y_1), \ldots, f(x_0, x_n, y_n)) = \\
&= (p(f(x_0\psi, x_1\psi, y_1\psi), \ldots, f(x_0\psi, x_n\psi, y_n\psi))\psi^{-1} = \\
&= (p(f(\langle x_0, x_0\rangle, \langle x_1, x_0\rangle, \langle x_0, y_1\rangle), \ldots, f(\langle x_0, x_0\rangle, \langle x_n, x_0\rangle, \langle x_0, y_n\rangle)\psi^{-1} = \\
&= \langle p(f(x_0, x_1, x_0), \ldots, f(x_0, x_n, x_0)), p(f(x_0, x_0, y_1), \ldots, f(x_0, x_0, y_n))\rangle\psi^{-1} = \\
&= \langle p(x_1, \ldots, x_n), p(y_1, \ldots, y_n)\rangle\psi^{-1} = \\
&= \langle f(x_0, p(x_1, \ldots, x_n), x_0), f(x_0, x_0, p(y_1, \ldots, y_n))\rangle\psi^{-1} = \\
&= (f(\langle x_0, x_0\rangle, \langle p(x_1, \ldots, x_n), x_0\rangle, \langle x_0, p(y_1, \ldots, y_n)\rangle))\psi^{-1} = \\
&= (f(x_0\psi, (p(x_1, \ldots, x_n))\psi, (p(y_1, \ldots, y_n))\psi))\psi^{-1} = \\
&= f(x_0, p(x_1, \ldots, x_n), p(y_1, \ldots, y_n)).
\end{aligned}$$

Since (3) holds identically in $\mathscr{R}$, for $p=f$ and for any $x, y, z\in\mathbf{F}_\omega$ we get

$$\begin{aligned}
x+(y+z) &= f(f(x_0, x_0, x_0), f(x_0, x, x_0), f(x_0, y, z)) = \\
&= f(x_0, f(x_0, x, y), f(x_0, x_0, z)) = (x+y)+z,
\end{aligned}$$

i.e. the addition is associative. Commutativity of addition may be checked analogously. Finally, again by (3),

$$x+f(x, x_0, x_0) = f(x_0, f(x_0, x, x_0), f(x, x_0, x_0)) =$$
$$= f(f(x_0, x_0, x), f(x_0, x, x_0), f(x_0, x_0, x_0)) = f(x, x, x_0) = x_0,$$

showing that $f(x, x_0, x_0)$ is the additive inverse of x in $\mathbf{F}_\omega$.

Let us consider the set R of all unary algebraic functions over $\mathbf{F}_\omega$ which involve no constants unless x_0. For each such function τ there exists a binary polynomial t over $\mathbf{F}_\omega$ such that $x\tau = t(x_0, x)$ for any $x \in \mathbf{F}_\omega$. We define addition and multiplication on R as follows:

$$x(\tau_1+\tau_2) = x\tau_1 + x\tau_2, \quad x(\tau_1\tau_2) = (x\tau_1)\tau_2.$$

It may be seen immediately that R is closed under these operations; furthermore, addition is associative, commutative and invertible (namely, $x(-\tau) = f(t(x_0, x), x_0, x_0)$), while multiplication is associative, left distributive and has a unit element (namely, the identical function). Right distributivity follows from (3):

$$x((\tau_1+\tau_2)\tau_3) = (x\tau_1+x\tau_2)\tau_3 = t_3(x_0, f(x_0, t_1(x_0, x), t_2(x_0, x))) =$$
$$= f(x_0, t_3(x_0, t_1(x_0, x)), t_3(x_0, t_2(x_0, x))) = (x\tau_1)\tau_3 + (x\tau_2)\tau_3 = x(\tau_1\tau_3 + \tau_2\tau_3).$$

Thus, $\mathbf{R} = \langle R; +, \cdot \rangle$ is a ring with unit element. Let $\bar{R} = \{\bar{\varrho} \mid \varrho \in R\}$, and $\bar{\mathbf{R}} = \langle \bar{R}; +, \cdot \rangle$ a ring isomorphic to $\mathbf{R}$ under the one-to-one correspondence $\bar{\varrho} \leftrightarrow \varrho$.

To get the desired equivalence we show the existence of a weak isomorphism χ of the free affine module $\mathbf{G}$ over $\bar{\mathbf{R}}$ with countable generating set onto $\mathbf{F}_\omega$ which maps the free generating set of $\mathbf{G}$ onto that of $\mathbf{F}_\omega$. By Lemma 2, $\mathbf{G}$ may be given in the form $\mathbf{G} = \mathbf{F}^*$, where $\mathbf{F}$ is the free $\bar{\mathbf{R}}$-module with countable free generating set $\{x_1, x_2, \ldots\}$, and the free generators of $\mathbf{F}^*$ are $\{0, x_1, x_2, \ldots\}$; then each element of G can be written in the form $x_{i_1}\bar{\varrho}_1 + \cdots + x_{i_m}\bar{\varrho}_m$, where $\bar{\varrho}_1, \ldots, \bar{\varrho}_m$ are non-zero elements of $\bar{\mathbf{R}}$, and this representation is unique (empty sum is allowed).

Define now χ by $(x_{i_1}\bar{\varrho}_1 + \cdots + x_{i_m}\bar{\varrho}_m)\chi = x_{i_1}\varrho_1 + \cdots + x_{i_m}\varrho_m$ (at the right hand side, x_{i_k} are free generators of $\mathbf{F}_\omega$, addition and "scalars" are the above-defined algebraic functions on $\mathbf{F}_\omega$). Furthermore, let $0\chi = x_0$. Then χ maps the free generating set of $\mathbf{G}$ onto that of $\mathbf{F}_\omega$. Observe that χ is one-to-one; indeed, if

$$x_{i_1}\varrho_1 + \cdots + x_{i_m}\varrho_m = x_{j_1}\sigma_1 + \cdots + x_{j_n}\sigma_n \tag{4}$$

holds in $\mathbf{F}_\omega$, then (4) (which is a short form for the equality of two elements of $\mathbf{F}_\omega$, whose full expression involves the polynomials $f, r_1, \ldots, r_m, s_1, \ldots, s_n$ and the free generators $x_0, x_{i_1}, \ldots, x_{i_m}, x_{j_1}, \ldots, x_{j_n}$) holds identically in $\mathscr{R}$. Replace all x_{i_k} and x_{j_l}, except x_{i_1} by x_0; then, using idempotency of all r_k and s_l we get $r_1(x_0, x_{i_1}) = x_0$ unless $x_{i_1} = x_{j_l}$ for some l. But $r_1(x_0, x_{i_1}) = x_0$ holds also identically in $\mathscr{R}$, whence it follows $\varrho_1 = 0$ in $\mathbf{R}$, a contradiction. Thus we conclude that $x_{i_1} = x_{j_l}$ and $r_1(x_0, x_{i_1}) =$

$=s_l(x_0, x_{j_l})=s_l(x_0, x_{i_1})$, whence $\varrho_1=\sigma_l$ in $\mathbf{R}$. Now a trivial induction shows that the two sides of (4) are the same.

To prove that χ is onto we use (3) in the form $p((x_1+y_1), \ldots, (x_n+y_n))=$ $=p(x_1, \ldots, x_n)+p(y_1, \ldots, y_n)$ (i.e., addition in $\mathbf{F}_\omega$ commutes with all polynomials). Any element of $\mathbf{F}_\omega$ can be written in the form $p(x_0, \ldots, x_t)$; but

$$p(x_0, \ldots, x_t) = p(x_0+x_0+\cdots+x_0, x_0+x_1+x_0+\cdots+x_0, \ldots, x_0+\cdots+x_0+x_t) =$$
$$= p(x_0, x_0, \ldots, x_0)+p(x_0, x_1, x_0, \ldots, x_0)+\cdots+p(x_0, \ldots, x_0, x_t).$$

For any i $(1\leqq i\leqq t)$ there exists a binary polynomial p_i such that

$$p(x_0, \ldots, x_0, x_i, x_0, \ldots, x_0) = p_i(x_0, x_i).$$

Hence

$$p(x_0, \ldots, x_t) = p_1(x_0, x_1)+\cdots+p_t(x_0, x_t) = x_1\pi_1+\cdots+x_t\pi_t = (x_1\bar{\pi}_1+\cdots+x_t\bar{\pi}_t)\chi,$$

where the unary algebraic functions π_i in $\mathbf{R}$ are defined by $x\pi_i=p_i(x_0, x)$.

To complete the proof, we need a one-to-one correspondence ζ between all polynomials $\mathbf{G}$ and $\mathbf{F}_\omega$ with the property that for any n-ary polynomial q of $\mathbf{G}$ and for arbitrary $y_1, \ldots, y_n\in G$ the equality

$$(5) \qquad (q(y_1, \ldots, y_n))\chi = (q\zeta)(y_1\chi, \ldots, y_n\chi)$$

holds. Since polynomials of $\mathbf{G}$ are the same as (fundamental) operations, every polynomial of $\mathbf{G}$ is of the form $(\bar{\varrho}_1, \ldots, \bar{\varrho}_n)$, where $\bar{\varrho}_1, \ldots, \bar{\varrho}_n\in\bar{R}$, $\sum_i\bar{\varrho}_i=1$. Let $((\bar{\varrho}_1, \ldots, \bar{\varrho}_n)\zeta)(z_1, \ldots, z_n)=z_1\varrho_1+\cdots+z_n\varrho_n$. There is an n-ary algebraic function of $\mathbf{F}_\omega$ on the right side; we show that in fact it is a polynomial. Since $\mathscr{R}$ is Hamiltonian, the subalgebra $\mathbf{H}$ of $\mathbf{F}_\omega$ generated by $\{x_1, \ldots, x_n\}$ is a block for some congruence θ of $\mathbf{F}_\omega$. Then $x_1\varrho_1+\cdots+x_n\varrho_n\equiv x_1\varrho_1+\cdots+x_1\varrho_n=x_1(\theta)$, whence $x_1\varrho_1+\cdots+x_n\varrho_n\in\mathbf{H}$. Thus, there exists an n-ary polynomial r of $\mathbf{F}_\omega$ such that $x_1\varrho_1+\cdots+x_n\varrho_n=r(x_1, \ldots, x_n)$. Hence $z_1\varrho_1+\cdots+z_n\varrho_n=r(z_1, \ldots, z_n)$ follows for any $z_1, \ldots, z_n\in F_\omega$.

To show that ζ is onto and one-to-one we may proceed similarly as in the case of χ, but we must take into consideration that now the ϱ_i may equal 0. Finally we prove (5) for $q=(\bar{\varrho}_1, \ldots, \bar{\varrho}_n)$ and arbitrary elements $y_i=x_1\bar{\tau}_{i1}+\cdots+x_t\bar{\tau}_{it}$ $(i=1, \ldots, n)$ of $\mathbf{G}$. We have

$$(q(y_1, \ldots, y_n))\chi = (x_1(\bar{\tau}_{11}\bar{\varrho}_1+\cdots+\bar{\tau}_{n1}\bar{\varrho}_n)+\cdots+x_t(\bar{\tau}_{1t}\bar{\varrho}_1+\cdots+\bar{\tau}_{nt}\bar{\varrho}_n))\chi =$$
$$= x_1(\tau_{11}\varrho_1+\cdots+\tau_{n1}\varrho_n)+\cdots+x_t(\tau_{n1}\varrho_1+\cdots+\tau_{nt}\varrho_n) =$$
$$= (x_1\tau_{11}+\cdots+x_t\tau_{1t})\varrho_1+\cdots+(x_1\tau_{n1}+\cdots+x_t\tau_{nt})\varrho_n = (q\zeta)(y_1\chi, \ldots, y_n\chi),$$

which was needed.

Call a variety $\mathscr{A}$ Abelian if in all algebras of $\mathscr{A}$ any two operations commute (i.e., for any m-ary g and n-ary h,

$$g(h(x_{11}, \ldots, x_{1n}), \ldots, h(x_{m1}, \ldots, x_{mn})) = h(g(x_{11}, \ldots, x_{m1}), \ldots, (x_{1n}, \ldots, x_{mn}))$$

is an identity in $\mathscr{A}$).

Theorem 2. *A variety $\mathscr{R}$ is equivalent to $\mathscr{A}(\mathbf{R})$ for some commutative ring* $\mathbf{R}$ *if and only if $\mathscr{R}$ is Abelian and satisfies condition* $(*)$.

Proof. On the base of Theorem 1, necessity is obvious from the description of operations on affine modules and the definition of Abelian varieties. Let now $\mathscr{R}$ be an Abelian variety satisfying $(*)$. With notations used in the proof of Theorem 1, we have to show that the ring $\mathbf{R}$ of all unary algebraic functions on $\mathbf{F}_\omega$ involving no other constants than x_0, is commutative. Let $\varrho_1, \varrho_2 \in \mathbf{R}$; then for any $x \in F_\omega$,

$$x(\varrho_1\varrho_2) = r_2(x_0, r_1(x_0, x)) = r_2(r_1(x_0, x_0), r_1(x_0, x)) =$$
$$= r_1(r_2(x_0, x_0), r_2(x_0, x)) = r_1(x_0, r_2(x, x)) = x(\varrho_2\varrho_1),$$

i.e., $\varrho_1\varrho_2 = \varrho_2\varrho_1$.

Theorem 3. *A variety $\mathscr{R}$ is equivalent to the variety of all affine modules (=affine spaces) over a field if and only if $\mathscr{R}$ is Abelian, equationally complete and satisfies condition* $(*)$.

Proof. In view of the preceding theorems, necessity is implied by Givant's result mentioned in the introduction. Let now $\mathscr{R}$ be an equationally complete Abelian variety satisfying $(*)$. It is enough to show that the function ring $\mathbf{R}$ is simple. In other words, we need the following fact: if a commutative ring with unit element, say $\mathbf{P}$, has a proper ideal $\mathbf{J}$, then the variety $\mathscr{P}$ of all affine modules over $\mathbf{P}$ has a proper (non-trivial) subvariety.

We may assume that $\mathbf{P}/\mathbf{J}$ is not isomorphic to $\mathbf{P}$. Let $\tilde{\pi}$ denote that block of the congruence determined by $\mathbf{J}$ which contains $\pi(\in P)$. With any affine module over $\mathbf{P}/\mathbf{J}$ we can associate an affine module over $\mathbf{P}$ with the same base set by defining the operations as follows: $(\pi_1, \ldots, \pi_k)(y_1, \ldots, y_k) = (\tilde{\pi}_1, \ldots, \tilde{\pi}_k)(y_1, \ldots, y_k)$. Applying the closure operators $\mathbf{S}$, $\mathbf{H}$ and $\mathbf{P}$, one can easily check that the affine modules over $\mathbf{P}$ obtained by such a way form a subvariety $\mathscr{P}'$ of $\mathscr{P}$. Moreover, $\mathscr{P}'$ is equivalent to the variety of all affine modules over $\mathbf{P}/\mathbf{J}$, whence, especially, follows that $\mathscr{P}'$ is non-trivial. Finally, $\mathscr{P}=\mathscr{P}'$ implies that the variety of all affine modules over $\mathbf{P}$ is equivalent to the variety of all affine modules over $\mathbf{P}/\mathbf{J}$. The following theorem shows that this is not the case, and thus $\mathscr{P}'$ is a proper subvariety of $\mathscr{P}$, qu.e.d.

Theorem 4. *If, for any rings* $\mathbf{R}_1$ *and* $\mathbf{R}_2$, $\mathscr{A}(\mathbf{R}_1)$ *and* $\mathscr{A}(\mathbf{R}_2)$ *are equivalent, then* $\mathbf{R}_1$ *and* $\mathbf{R}_2$ *are isomorphic.*

Proof. For $i=1, 2$; $\mathbf{R}_i$ when considered as an $\mathbf{R}_i$-module is a free $\mathbf{R}$-module with the free generator 1. Lemma 2 implies that $\mathbf{R}_i^*$ — as an affine module over $\mathbf{R}_i$ — is free in $\mathscr{A}(\mathbf{R}_i)$ with the free generating set $\{0, 1\}$. As $\mathscr{A}(\mathbf{R}_1)$ is equivalent to $\mathscr{A}(\mathbf{R}_2)$ there exists a weak isomorphism χ of $\mathbf{R}_1^*$ onto $\mathbf{R}_2^*$ such that $0\chi=0$, $1\chi=1$, with correspondence of polynomials ζ. Let, especially, $(\varrho, 1-\varrho)\zeta=(\varrho', 1-\varrho')$ for any binary polynomial $(\varrho, 1-\varrho)$ of $\mathbf{R}_1^*$. Define the mapping $\varphi: R_1 \to R_2$ by $\varrho\varphi = \varrho'$. Since ζ is one-to-one and onto, the same is valid for φ. We show that φ is an isomorphism of $\mathbf{R}_1$ onto $\mathbf{R}_2$; for this aim, it suffices to show that φ is homomorphic.

First we prove

$$(1, -1, 1)\zeta = (1, -1, 1). \tag{6}$$

Let $(1, -1, 1)\zeta = (\alpha, \beta, \gamma)$; then using (5) for $q=(1, -1, 1)$ and $y_1=1$, $y_2=y_3=0$, we obtain $\alpha=1$. Similarly we get $\beta=-1$, $\gamma=1$.

Now from (5) and (6) it follows

$$(x+y)' = \big((x+y)', 1-(x+y)'\big)(1, 0) = \big((x+y, 1-(x+y))(1, 0)\big)\chi =$$
$$= \big((1, -1, 1)((x, 1-x)(1, 0), 0, (y, 1-y)(1, 0))\big)\chi =$$
$$= (1, -1, 1)\big(((x, 1-x)(1, 0))\chi, 0\chi, ((y, 1-y)(1, 0))\chi\big) =$$
$$= (1, -1, 1)\big((x', 1-x')(1, 0), 0, (y', 1-y')(1, 0)\big) = x'+y'.$$

Finally, for any $x, y \in R_1$ we have $(xy)\chi = \big(xy+0(1-y)\big)\chi = (y', 1-y')(x\chi, 0) = (x\chi)y'$, whence

$$(xy)' = (1\chi)(xy)' = (1xy)\chi = (1x)\chi \cdot y = (1\chi)x'y' = x'y',$$

completing the proof.

Note that the proof of theorem 3 indicates also the following result: A variety of form $\mathscr{A}(\mathbf{R})$ is equationally complete if and only if $\mathbf{R}$ is a simple ring.

References

[1] G. Grätzer, *Universal Algebra,* Van Nostrand (1968).
[2] S. Mac Lane—G. Birkhoff, *Algebra,* Macmillan (1967).
[3] A. I. Mal'cev, *The matemathematics of algebraic systems,* North-Holland (1971).
[4] B. Csákány, On the equivalence of certain classes of algebraic systems, *Acta Sci. Math.,* **23** (1962), 46—57.
[5] B. Csákány, Characterizations of regular varieties, *Acta Sci. Math.,* **31** (1970), 187—189.
[6] S. R. Givant, A representation theorem for universal classes of algebras in which all member are free, *Notices AMS,* **19**: 7 (1972), A—767.
[7] A. Goetz, On weak isomorphisms and weak homomorphisms of abstract algebras, *Coll. Math.,* **14** (1966), 163—167.
[8] L. Klukovits, Hamiltonian varieties of universal algebras, *Acta Sci. Math.,* **37** (1975), 11—15.
[9] F. Ostermann—J. Schmidt, Der baryzentrische Kalkül als axiomatische Grundlage der affinen Geometrie, *J. reine angew. Math.,* **224** (1966), 44—57.

(Received May 18, 1973)